UNIT-V

Game Theory

- Business Situation involves competition.
- Effective decision making is plays a vital role in such situations.

Decision making situations:

1. Deterministic - The problem data representing the situation are constants.
2. Probabili~~gt~~tic - Do not remain constant. Vary due to chance and represent as some probability distribution.
3. Uncertainity - Data subjected to variance and it is not in the probability distribution.

Pure Strategy:

If a player selects a particular strategy with a probability of 1, then that strategy is known as pure strategy. Ignore the remaining strategies.

Mixed Strategy:

If a player follows more than one strategy, then that strategy is known as mixed strategy. But the probability will be less than one.

Maximini Principle:

The Principle maximizes the minimum guaranteed gains of a player A. The minimum gains with respect to different alternatives of 'A'. The maximum of these minimum gains is known as maximini value.

Minimax Principle:

The principle minimizes the maximum losses. with respect to different alternatives of player B. The minimum of these maximum losses is known as minimax value.

Saddle Point:

In a game, if the maximini Value is equal to the minimax value, then the game is said to have a Saddle point. The intersecting cell corresponding to these values is known as saddle point.

If the game has a Saddle Point, then each player has a pure Strategy.

Example:

		Player B 1	2	3	Row minimum
	1	20	15	22	15
Player A	2	$\boxed{35}$	45	40	35 — Maximin
	3	18	20	25	18
Column maximum		35	45	40	

Minimax

Minimax = 35 ; Maximini = 35

∴ Saddle Point = (35, 35)

Value of the game is $\boxed{V = 35}$

Game with mixed Strategy: (2×2 matrix)

$$\begin{array}{cc|cc|c} & & \multicolumn{2}{c|}{B} & \\ & & 1 & 2 & \text{oddments} \\ \hline A & 1 & a & b & |c-d| \\ & 2 & c & d & |a-b| \\ \hline \text{oddments} & & |b-d| & |a-c| & \end{array}$$

$$p_1 = \frac{|c-d|}{|a-b| + |c-d|} \qquad q_1 = \frac{|b-d|}{|b-d| + |a-c|} \qquad V = \frac{a|c-d| + c|a-b|}{|a-b| + |c-d|}$$

$$p_2 = \frac{|a-b|}{|a-b| + |c-d|} \qquad q_2 = \frac{|a-c|}{|b-d| + |a-c|}$$

Example:

Solve the following 2×2 game

$$\begin{array}{c|cc} & & B \\ & 1 & 2 \\ \hline A\ 1 & 6 & 9 \\ 2 & 8 & 4 \end{array}$$

Same to arithmetic method

Solution:

$$\begin{array}{cc|cc|l} & & B & & \\ & & 1 & 2 & \text{Row minimum} \\ \hline A & 1 & 6 & 9 & 6 - \text{Maximin} \\ & 2 & 8 & 4 & 4 \\ \hline & \text{Column maximum} & 8 & 9 & \\ & & \text{minimax} & & \end{array}$$

The Pay off matrix has no Saddle Point. Therefore go for mixed Strategy.

$$\begin{array}{cc|cc|l} & & B & & \\ & & 1 & 2 & \text{oddments} \\ \hline A & 1 & 6 & 9 & 4\ (c-d) \\ & 2 & 8 & 4 & 3\ (a-b) \\ \hline & \text{oddments} & 5 & 2 & \\ & & (b-d) & (a-c) & \end{array}$$

$$P_1 = \frac{4}{4+3} = \frac{4}{7}$$

$$P_2 = \frac{3}{4+3} = \frac{3}{7}$$

$$q_1 = \frac{5}{5+2} = \frac{5}{7}$$

$$q_2 = \frac{2}{5+2} = \frac{2}{7}$$

$$V = \frac{6(4) + 8(3)}{3+4}$$

$$\Rightarrow V = \frac{48}{7}$$

Strategy of A is $\left(\frac{4}{7}, \frac{3}{7}\right)$ and $V = \frac{48}{7}$

B is $\left(\frac{5}{7}, \frac{2}{7}\right)$

Arithmatic method: [n×n matrix]

① Solve the following 3×3 game.

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 3 & -1 & -3 \\ -3 & 3 & -1 \\ -4 & -3 & 3 \end{bmatrix} \end{array}$$

Step-①:

Column matrix,
$$C = \begin{array}{c} \begin{array}{cc} 3-(-3) & -1-(-3) \end{array} \\ \begin{bmatrix} 4 & 2 \\ -6 & 4 \\ -1 & -6 \end{bmatrix} \end{array}$$

Step-②:

Row matrix,
$$R = \begin{bmatrix} 6 & -4 & -2 \\ 1 & 6 & -4 \end{bmatrix}$$

Step-③:

magnitude,
$$|C_1| = \begin{vmatrix} -6 & 4 \\ -1 & -6 \end{vmatrix} = 36-(-4) = 40$$

$$|C_2| = \begin{vmatrix} 4 & 2 \\ -1 & -6 \end{vmatrix} = -22$$

$$|C_3| = \begin{vmatrix} 4 & 2 \\ -6 & 4 \end{vmatrix} = 28$$

$$|R_1| = \begin{vmatrix} -4 & -2 \\ 6 & -4 \end{vmatrix} = 28$$

$$|R_2| = \begin{vmatrix} 6 & -2 \\ 1 & -4 \end{vmatrix} = -22$$

$$|R_3| = \begin{vmatrix} 6 & -4 \\ 1 & 6 \end{vmatrix} = 40$$

Step ④: Augmented Payoff matrix

$$\begin{array}{cc|ccc|c} & & & B & & \text{Row oddments} \\ & & 3 & -1 & -3 & 40 \\ A & & -3 & 3 & -1 & 22 \\ & & -4 & -3 & 3 & 28 \\ \text{Column oddments} & & 28 & 22 & 40 & \boxed{90} \end{array}$$

Step-⑤: (Strategy)

Row Player A : $\left(\frac{40}{90}, \frac{22}{90}, \frac{28}{90}\right)$

: $\left(\frac{4}{9}, \frac{11}{45}, \frac{14}{45}\right)$

Column player B : $\left(\frac{28}{90}, \frac{22}{90}, \frac{40}{90}\right)$

: $\left(\frac{14}{45}, \frac{11}{45}, \frac{4}{9}\right)$

Step-⑥: (Value)

$$V = \frac{4}{9} \times 3 + \frac{11}{45} \times -3 + \frac{14}{45} \times -4$$

$$\Rightarrow \boxed{V = -\frac{29}{45}}$$

$$V = \frac{14}{45} \times 3 + \frac{11}{45} \times -1 + \frac{4}{9} \times -3$$

$$\Rightarrow \boxed{V = \frac{-29}{45}}$$

(2) For the payoff matrix given below, decide the optimum strategies for A & B.

$$\begin{array}{cc} & B \\ & \begin{array}{cc} 1 & 2 \end{array} \\ A \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 6 & 9 \\ 8 & 4 \end{bmatrix} \end{array}$$

Solution:

$$C = \begin{bmatrix} -3 \\ +4 \end{bmatrix} \qquad R = \begin{bmatrix} -2 \\ +5 \end{bmatrix}$$

$|C_1| = +4 \qquad |R_1| = +5$

$|C_2| = -3 \qquad |R_2| = -2$

Augmented Payoff matrix }

$$\begin{array}{cc} & B \\ & \begin{array}{cc} 1 & 2 \end{array} \\ A \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 6 & 9 \\ 8 & 4 \end{bmatrix} \begin{array}{c} +4 \\ 3 \end{array} \\ & \begin{array}{ccc} 5 & 2 & \boxed{7} \end{array} \end{array}$$

Strategy for B : $\left(\frac{5}{7}, \frac{2}{7}\right)$

Strategy for A : $\left(\frac{4}{7}, \frac{3}{7}\right)$

Value, $V = \frac{4}{7} \times 6 + \frac{3}{7} \times 8$

$\Rightarrow \boxed{V = \frac{48}{7}}$

$V = \frac{5}{7} \times 6 + \frac{2}{7} \times 9$

$\Rightarrow \boxed{V = \frac{48}{7}}$

③ Solve Game.

$$\begin{array}{ccc} & & \begin{array}{ccc} & B & \\ I & II & III \end{array} \\ A & \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 6 & 1 & 6 \end{bmatrix} \end{array}$$

Solution:

$$C = \begin{bmatrix} -6 & 5 \\ 4 & -5 \\ 5 & -5 \end{bmatrix} \qquad R = \begin{bmatrix} -5 & 5 & -5 \\ 0 & 1 & 1 \end{bmatrix}$$

$$C_1 = \begin{vmatrix} 4 & -5 \\ 5 & -5 \end{vmatrix} = 5 \qquad R_1 = \begin{vmatrix} 5 & -5 \\ 1 & 1 \end{vmatrix} = 10$$

$$C_2 = \begin{vmatrix} -6 & 5 \\ 5 & -5 \end{vmatrix} = 5 \qquad R_2 = \begin{vmatrix} -5 & -5 \\ 0 & 1 \end{vmatrix} = -5$$

$$C_3 = \begin{vmatrix} -6 & 5 \\ 4 & -5 \end{vmatrix} = 10 \qquad R_3 = \begin{vmatrix} -5 & 5 \\ 0 & 1 \end{vmatrix} = -5$$

Augmented matrix,

				oddments
	1	7	2	5
	6	2	7	5
	6	1	6	10
oddments	10	5	5	$\boxed{20}$

$\therefore$ optimum Strategies are:

Player A : $\left(\frac{5}{20}, \frac{5}{20}, \frac{10}{20}\right)$

: $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$

Player B : $\left(\frac{10}{20}, \frac{5}{20}, \frac{5}{20}\right)$

: $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$

$$\text{Value} = \frac{1}{4} \times 1 + \frac{1}{4} \times 6 + \frac{1}{2} \times 6$$

$$\boxed{V = \frac{19}{4}}$$

Dominance Property:

In some games, it is possible to reduce the size of the payoff matrix by eliminating the redundant rows (or columns). If a game has such redundant rows (or columns), those rows are dominated by other rows. Such property is known as dominance property.

Dominance Property for Rows:

a) If all the entries in a Row (X) are greater than (or) equal to the corresponding entries of another Row (Y), then Row (Y) is dominated by Row(X). Under such situation Row (Y) is deleted.

b) If sum of the any two rows (Row X + Row Y) is greater than (or) equal to the third row (Row Z) then Row Z is dominated by Row X + Y. Under such situation Row Z is deleted.

$$\text{Row } X \geq \text{Row } Y \Rightarrow \text{Row } Y \text{ is deleted}$$

$$\text{Row } X + \text{Row } Y \geq \text{Row } Z \Rightarrow \text{Row } Z \text{ is deleted}$$

Dominance Property for Columns:

$$\text{Column } X \leq \text{Column } Y \Rightarrow \text{Column } Y \text{ is deleted}$$

$$\text{Column } X + \text{Column } Y \leq \text{Column } Z \Rightarrow \text{Column } Z \text{ is deleted}$$

① Solve the following 3×3 game.

$$\begin{array}{cc} & \begin{array}{ccc} & B & \\ I & II & III \end{array} \\ A \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 6 & 1 & 6 \end{bmatrix} \end{array}$$

<u>Solution:</u>

		B I	B II	B III	Minimum
A	I	1	7	2	1
	II	6	2	7	2 Maximini
	III	6	1	6	1
	Maximum	6	7	7	
		Minimax			

Maximini ≠ Minimax

∴ There is no saddle point.

Check for dominance Property.

Row III is dominated by Row II. ∴ Row III is removed.

$$\begin{array}{cc} & \begin{array}{ccc} I & II & III \end{array} \\ \begin{array}{c} I \\ II \end{array} & \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \end{bmatrix} \end{array}$$

Column III is dominated by Column I. ∴ Column III is removed.

$$\begin{array}{cc} & \begin{array}{cc} & B \\ I & II \end{array} \\ A \begin{array}{c} I \\ II \end{array} & \begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix} \end{array}$$

Column matrix, $C = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$

Row matrix, $R = \begin{bmatrix} -5 & 5 \end{bmatrix}$

Magnitude,

$|C_1| = 4$ $\qquad$ $|R_1| = 5$

$|C_2| = 6$ $\qquad$ $|R_2| = 5$

Augmented Payoff matrix,

$$\begin{array}{ccccc} & & & B & & \\ & & \text{I} & \text{II} & \\ A & \text{I} & 1 & 7 & 4 \\ & \text{II} & 6 & 2 & 6 \\ & & 5 & 5 & \boxed{10} \end{array}$$

Optimum Strategy,

Player A : $\left(\frac{4}{10}, \frac{6}{10}, 0\right) \Rightarrow \left(\frac{2}{5}, \frac{3}{5}, 0\right)$

Player B : $\left(\frac{5}{10}, \frac{5}{10}, 0\right) \Rightarrow \left(\frac{1}{2}, \frac{1}{2}, 0\right)$

Value of game,

$$V = \frac{2}{5} \times 1 + \frac{3}{5} \times 6$$

$$\Rightarrow \boxed{V = 4}$$